Synthesis of the Types of Optimal Transfers between Hyperbolic Asymptotes

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Many space missions currently being studied and undertaken require mid-mission swingbys for trajectory modification, or investigatory probe passes of one or more planets enroute to the final mission goal. Because such missions are still severely weight-limited, the problem of performing these maneuvers in a fuel-optimal manner is of high interest. This paper solves the problem of determining the type of minimum characteristic velocity transfer between two given hyperbolic excess velocity vectors associated with a real planet. The radius of the planet introduces a maximum natural turn angle, achieved by grazing passage, for any given hyperbolic excess velocity. For given approach and departure asymptotic velocities, this leads to an optimal deviation angle which neatly divides the problem into two categories. The first is the case for which the turn angle required by the given asymptotes is less than or equal to the optimal deviation angle. The resulting optimal transfers are of five types. In boundary condition space, by far the largest region requires either single impulse transfers or transfers through the parabolic level requiring two finite impulses. The second category is that where the turn angle required by the given asymptotes is greater than the optimal deviation angle. The resulting optimal transfers are of six types, all grazing. Two-impulse transfers and transfers through the parabolic level play a much larger role than in the first case. This paper presents graphically the type of optimal transfer for all possible arrival and departure asymptotic conditions. This knowledge of the type of transfer which yields the optimal transfer greatly reduces the effort required to determine the optimal maneuver.

Introduction

THE original interest in the orbit transfer problem was in the transfer between elliptical orbits and in escape from a planet. For the usual transfer problem of minimizing the characteristic velocity of the maneuver about a point mass planet with no restriction on the time elapsed during the transfer, it was natural to neglect hyperbolic trajectories. This is because between hyperbolic trajectories under these circumstances, the transfer can always be accomplished with zero characteristic velocity.

Now, with active consideration of planetary flybys, multiple planet missions, and roundtrip swingbys under way, the transfer between hyperbolic trajectories becomes a natural problem. For a particular planetary pass, the obvious problem formulation is that a hyperbolic arrival velocity vector is given far from the planet and a desired hyperbolic departure velocity vector far from the planet is to be attained. A transfer between these states is to be determined. The normal optimization criterion in this case is to find the minimum characteristic velocity maneuver which accomplishes the desired transfer.

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As pointed out, if both the assumptions of a time open transfer and a point mass planet are made, the problem becomes trivial and the answer unrealistic. to obtain a well-posed problem, one of these assumptions must be altered. If the point mass planet is replaced by a real planet with finite radius, the problem becomes reasonable and the answers interesting, even if the time open assumption is retained.

The givens of the problem[1,2] are the gravitational constant μ and radius R of the planet, magnitudes of the hyperbolic excess velocities of arrival and departure V_1 and V_2, and the angle A between the arrival and departure asymptotes. The required is the minimum characteristic velocity transfer between the arrival conditions and departure conditions, subject to the physical constraint of the spherical planet.

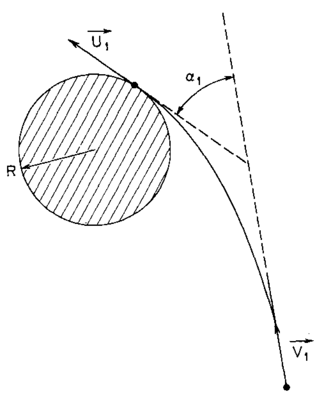


Fig. 1 Half the maximum natural turn angle corresponding to the given planet and a given velocity at infinity, V_1.

The periapsis position of the hyperbola associated with a given c velocity can be moved arbitrarily both in direction and radial distance at negligible fuel cost. This has two important consequences. The problem as stated becomes planar, and the turn angle may be manipulated.

The natural turn angle is simply the scattering angle caused by the vehicle's encounter with the planet. The maximum natural turn angle is that obtained by moving the periapsis radial distance to R, with the trajectory then a grazing hyperbola. Half the maximum natural turn angle is designated α; Fig. 1. It is convenient to introduce L, the escape velocity at the surface of the planet (which can replace μ in specifying the planet) and U, the perivelocity at level R on a trajectory of hyperbolic excess velocity V. The optimal deviation angle corresponding to a given V_1 and V_2 is α_1 plus α_2. The significance of this angle is evident from consideration of the case in which the required turn angle A is equal to the optimal deviation angle. Then the optimal transfer is composed of two half branches of grazing hyperbolas with a common periapse; Fig. 2. It requires a single impulse, at periapse, of magnitude $|U_2-U_1|$. If $A\neq\alpha_1+\alpha_2$, then the characteristic velocity of the transfer must be greater than this.

There is one other special case which deserves mention. If V_1 and V_2 are the asymptotic velocities of the same hyperbola, and if this hyperbola does not intersect the forbidden sphere, then no artifical velocity change is needed and the optimal transfer is the free transfer with C equal to zero. Considering that one is able to move the periapsis at no cost, the free transfer is the case where V_1 equals V_2, α_1 equals α_2, and $A<2\alpha_1$.

Thus, the free transfer is available only when the turn angle demanded, A, is less than or equal to the optimal deviation angle of the problem, $\alpha_1+\alpha_2$. This brings one to the realization that the optimal deviation angle neatly divides the basic problem into two distinct classes[3]:

$$A\leq\alpha_1+\alpha_2 \tag{1}$$

and

$$A>\alpha_1+\alpha_2 \tag{2}$$

This paper deals with both cases. In general, the maximum principle as propounded by Pontryagin[4] is applied. Applied to orbit transfer optimization, the theory was developed independently in a restricted form by Contensou[5] and Busemann.[6] Lawden[7] used this approach on optimal trajectory problems first, and so highly developed the theory that the "Lawden primer vector" bears his name. The following sections will summarize the results of the investigation.[2]

Transfers with Turn Angle Less Than the Optimal Deviation Angle

It will be assumed, by using reversibility if necessary, that $V_2\leq V_1$. The optimal transfers are of five different types: 1) Type F: nongrazing transfer with one impulse at a finite distance; 2) Type F∞: nongrazing transfer with a first impulse at a finite distance and a second impulse at an infinite distance from the planet; 3) Type RF: grazing transfer corresponding to type F; 4) Type RF∞: grazing transfer corresponding to type F∞; 5) Type PNP: transfer through the parabolic level of energy, *par le niveau parabolique*. In the case $V_2\geq V_1$, the types F∞, RF, and RF∞ become ∞F, FR, and ∞FR.

Transfers of Type F

This case is easy to calculate using simple max-min theory; Fig. 3. To simplify the results, the following variables will be introduced:

$$\Sigma=45^\circ-A/4 \tag{3}$$

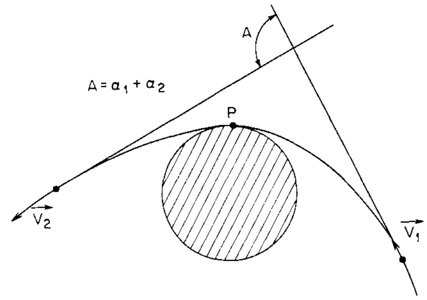


Fig. 2 The optimal deviation angle transfer.

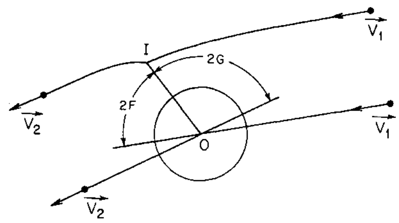


Fig. 3 The F transfer.

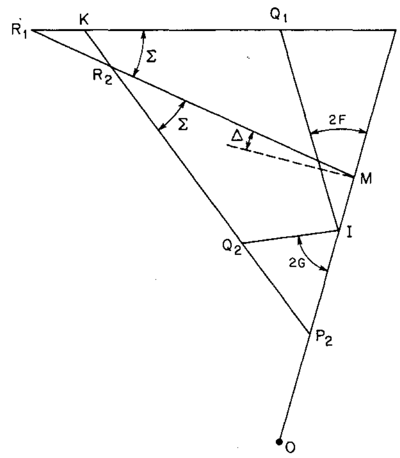


Fig. 4 The hodograph for an optimal F transfer.

$$\Delta=\text{arc tan}[(V_1-V_2/V_1+V_2)\tan\Sigma] \tag{4}$$

$$F=\Sigma-\Delta,\quad G=\Sigma+\Delta \tag{5}$$

where F and G determine the direction to the point I where the optimal impulse is applied. Using these variables, the distance from the center of attraction is given by

$$OI=\{R\,L^2\sin^2F/V_2^2\cos^2\Delta[2\cos^2\Sigma-\cos^2\Delta]\} \tag{6}$$

and the transfer is completely determined. The characteristic velocity of the transfer is

$$C=(V_1+V_2)\sin\Delta \tag{7}$$

The figure made at I by the velocities of arrival (V_1') and departure (V_2') for the optimal F transfer has many geometrical properties which will be very useful for the other types of transfers; Fig. 4.

$$IR_1=V_1' \quad IR_2=V_2' \quad IO_1=V_1 \quad IO_2=V_2 \quad IP_1=IQ_1$$

$$IP_2=IQ_2 \quad KR_1=KR_2 \quad P_1Q_1=P_2Q_2$$

$$P_1R_1=P_2R_2 \quad MP_1=MP_2$$

$$OMR_1=90^\circ+\Delta \quad C=R_1R_2=(V_1+V_2)\sin\Delta \tag{8}$$

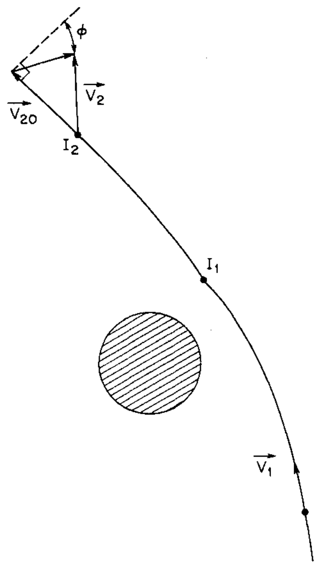


Fig. 5 The F∞ transfer.

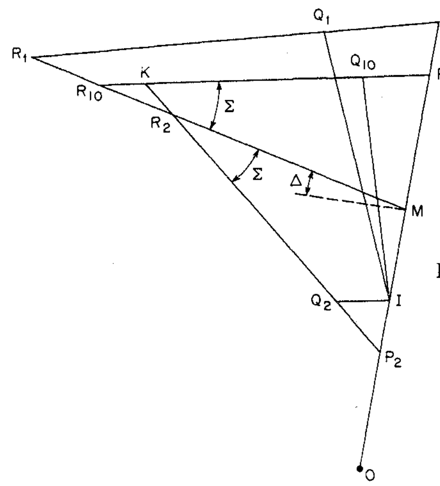


Fig. 6 The RF transfer.

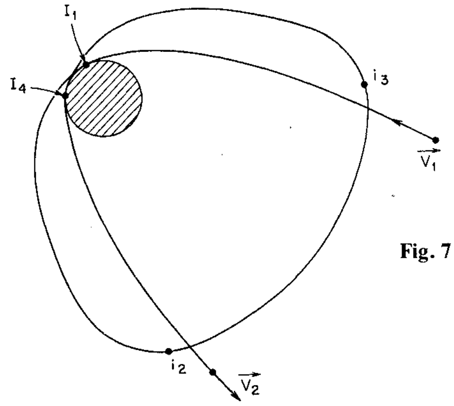


Fig. 7 The PNP transfer.

Transfers of Type F∞

If the impulse at infinity is suppressed, an optimal F-type transfer will be obtained, but a particular one, in which the primer vector of Lawden[7] has final length equal to one. This leads to an additional relation between Σ and Δ:

$$\Sigma = f(\Delta) = \arc\tan \left[\frac{t}{4+t^2} \, (4+4t^2 + (4+5t^2+10t^4)^{1/2}) \right] \tag{9}$$

with $t = \tan \Delta$ (Δ increases from 0°-28.99° when Σ increases from 0°-45°). The velocity V_{20} just before the second impulse is given by

$$V_{20} = V_1/15[6 \sin^2\Delta - 1 + 2(4 - 3\sin^2\Delta + 9\sin^4\Delta)^{1/2}] \tag{10}$$

The eccentricity e of the orbit lying between the two impulses is given by

$$(e^2-1)^{1/2} = (\sin 2\Sigma - \sin 2\Delta)/(2\cos^2\Sigma - \cos^2\Delta) \tag{11}$$

Thus, limits on e are $1 < e < 1.191$. Finally, the direction of the second impulse (at infinity) is located by the angle (ϕ_2) with the forward horizontal (Fig. 5). V_{20} is vertical, and V_2 is always smaller than V_{20}.

$$\cos \phi_2 = \frac{\sin \Delta \cos^3\Delta}{2 \cos \Sigma \sin(\Sigma - \Delta)} \tag{12}$$

$$\sin \phi_2 = \frac{1}{\sin(\Sigma-\Delta)} \left[\cos \Sigma - \sin \Sigma \sin \Delta \cos \Delta - \frac{\cos^2\Delta}{\cos \Sigma} (2\cos^2\Sigma - \cos^2\Delta) \right] \tag{13}$$

From Eq. (12) and the limits on Σ and Δ, ϕ_2 is limited to $0° < \phi_2 < 33.74°$ and $\pi - 4\Sigma = A_0 > A$. For an optimal F-type transfer, the final length of the primer vector must be less than or equal to one, hence we must have $\Sigma > f(\Delta)$ instead of $\Sigma = f(\Delta)$, and thus the equality in Eq. (10) is replaced by a "greater than or equal" sign.

RF Transfer

This transfer is similar to the F transfer except the vehicle grazes the forbidden sphere. This has the effect of dividing the trajectory into the two parts, before and after graze. The primer vector is analyzed separately on the two sections. A general result is that the graze must occur before the impulse (for the convention that $V_2 < V_1$). An analysis similar to that for the F transfer applies, and the illustration for the optimal configuration of the single impulse is given in Fig. 6 in which the points with the subscript 0 correspond to the unconstrained optimal F transfer. The direction of the impulse to change from the grazing hyperbolic branch to the final hyperbolic branch of the transfer must be equal to Δ for the unconstrained F transfer as shown in Fig. 6. The optimality of the transfer requires the point R_1 to be at a greater distance from M than the point R_{10}.

Transfers of Type RF∞

This transfer is a combination of the two previous cases: Σ and Δ must satisfy Eq. (9), V_1' and V_2' must verify the construction of Fig. 6 and the second impulse (at infinity) has a direction (ϕ_2) given by Fig. 5 and Eqs. (12) and (13).

PNP Transfer

The PNP transfer, or biparabolic transfer, is the adaptation to the case of a planet with finite radius of the six infinitesimal impulse transfer[8] of the unconstrained transfer problem. The PNP transfer is shown in the accompanying figure (Fig. 7). The vehicle is first put on a grazing trajectory by the customary infinitesimal impulse at infinity. At periapse, at the first graze, a tangential braking impulse of magnitude ($U_1 - L$) leads to the parabolic trajectory shown. At infinity, an infinitesimal impulse i_2 leads to a circular orbit which is traversed until the proper orientation is reached. Then a second infinitesimal impulse i_3 places the vehicle on the grazing parabolic trajectory which has a common periapse with the grazing hyperbola of the desired asymptotic velocity. At this common periapse, a tangential impulse I_4 of magnitude ($U_2 - L$) accelerates the vehicle and leaves it on the desired departure trajectory.

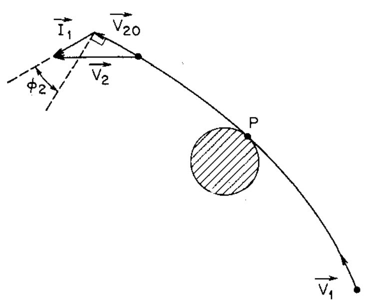


Fig. 8 The R∞ transfer.

The characteristic velocity of the PNP transfer is (U_1+U_2-2L), and is independent of turn angle A. It can also be demonstrated that where $\alpha_1+\alpha_2<A<180°-|\alpha_1-\alpha_2|$ the use of three dimensions may save one infinitesimal impulse between the two finite impulses. This leaves the cost unchanged. With the PNP transfer as an upper bound on the cost, and the maximum natural turn angle transfer as a lower bound, bounds for the cost of any optimal transfer, regardless of the turn angle demanded, are

$$|U_2-U_1|<C<U_1+U_2-2L \tag{14}$$

Greater than Optimal Deviation Transfers

Assume, by using reversibility if necessary, that $V_2>V_1$. These transfers are of six different types which are always grazing: 1) Type PNP: identical to that of the less than optimal deviation case and always optimal for $A=180°$; 2) Type R∞: transfer with one impulse at infinity on the side of the greater velocity; 3) Type ∞R∞: transfer with two impulses at infinity, one on either side of the grazing passage; 4) Type RF: transfer with one impulse at a finite distance after the grazing passage; 5) Type RF∞: transfer with two impulses after the grazing passage, one at a finite distance and the other at infinity; and 6) Type ∞RF: transfer with two impulses, one at infinity before the grazing passage and the other at a finite distance after the passage. If $V_2<V_1$ the types R∞, RF, RF∞, and ∞RF become ∞R, FR, ∞RF and RF∞.

Transfer of Type R∞

This transfer begins as a grazing hyperbola H and ends at infinity with the impulse $\boldsymbol{I}_1=\boldsymbol{V}_2-\boldsymbol{V}_{20}$; Fig. 8. The impulse is easy to calculate. The direction of $\boldsymbol{I}_1$ is located by the angle ϕ_2 with the forward horizontal and the optimality requires

$$\arctan\,[2V_1L^2/U_1(L^2+2V_1{}^2)]<\phi_2<\arctan\,(2^{1/2}/2) \tag{15}$$

or $0°<\phi_2<35.264°$.

Transfers of Type ∞R∞

The transfer begins by a finite impulse $\boldsymbol{I}_1$ at infinity leading to the grazing hyperbola H_1 and ends by a second impulse $\boldsymbol{I}_2$ at infinity on the other side; Fig. 9. The directions of the two impulses are located by the angles $(\pi-\phi_1)$ and ϕ_2, with the local forward horizontal (measured clockwise if H_1 is the counterclockwise direction and conversely). The optimality requires

$$\phi_2=\phi_1=\arctan\,[2V_{10}L^2/U_{10}(L^2+2V_{10}{}^2)] \tag{16}$$

Transfers of Type RF

Let us use Fig. 10, similar to Figs. 4 and 6. As in the "Less Than Optimal Deviation Angle Case" the theory of optimization of Pontryagin[4] leads to a very simple result: 1) the grazing passage must occur before the impulse; 2) the velocity $\boldsymbol{V}_1=\boldsymbol{IR}_1$ of arrival at the impulse, corresponds to a point R_1 nearer to M than R_2 with the velocity of departure given by $\boldsymbol{V}_2=\boldsymbol{IR}_2$. In this case it is possible to have $\Sigma=(180-A_0/4)$

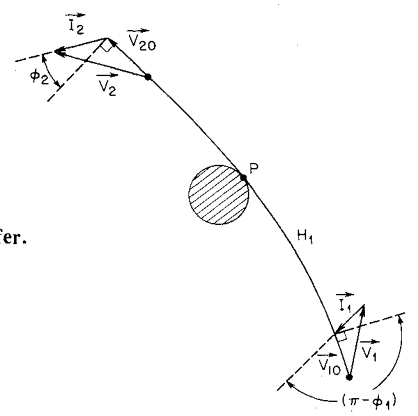


Fig. 9 The ∞R∞ transfer.

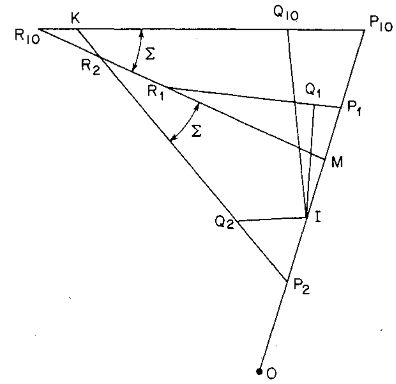


Fig. 10 The RF transfer.

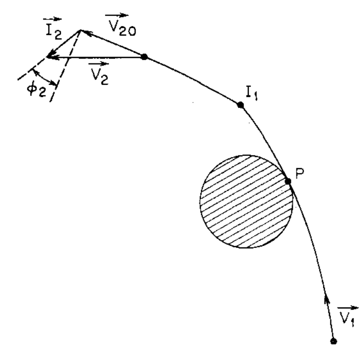


Fig. 11 The RF∞ transfer.

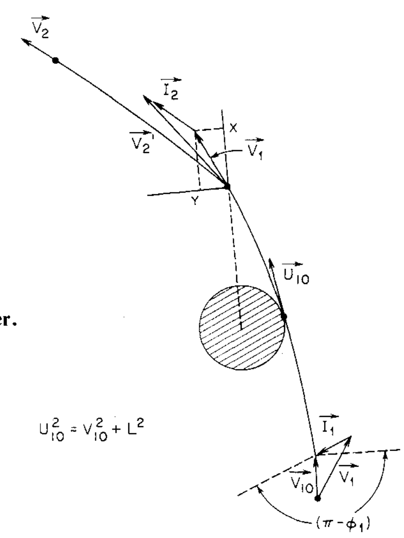
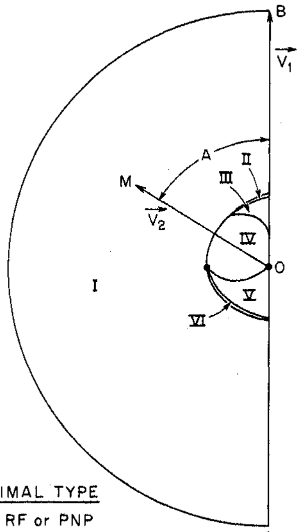


Fig. 12 The ∞RF transfer.

greater than 45° but $(2\cos^2\Sigma-\cos^2\Delta)$ remains positive and the relation $\Sigma>f(\Delta)$ is still valid, hence

$$0°<\Delta<\arcsin\,(\sqrt{3}/3)=35.264° \text{ and}$$

$$0°<3/3\Delta<\Sigma<\arccos\,(\sqrt{3}/3)=54.736° \tag{17}$$



M IN REGION	OPTIMAL TYPE
I	F or RF or PNP
II & VI	F∞ or RF or RF∞ or PNP
III & V	RF or RF∞ or PNP
IV	RF or PNP

Fig. 13 The classification of the optimal type of transfer in which the dependency on the scale L is suppressed. This figure applies only to cases for which $A < \alpha_I + \alpha_2$.

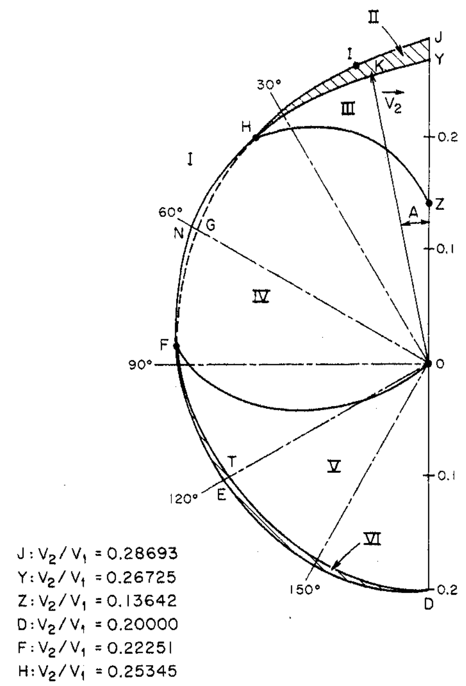


Fig. 14 Enlargement of the central portion of Fig. 13.

The impulse $R_I R_2$ at the point I is now an accelerating one instead of a decelerating one.

Transfers of Type RF∞

As n the case of type F∞, if one suppresses the impulse at infinity, one obtains another optimal transfer which is this time, of the "Greater Than Optimal Deviation Angle RF type." As in the F∞ case, the RF transfer obtained is a particular one. The final length of the primer vector of Lawden must be equal to one. Hence, Σ and Δ are bounded by Eq. (9).

Finally, the direction of the second impulse at infinity is located by the angle ϕ_2 with the local forward horizontal (Fig. 11), ϕ_2 being given by Eqs. (12) and (13). Hence, Σ being between 0° and 54.736°, we have

$$0° < \Delta < \phi_2 < 35.264° \tag{18}$$

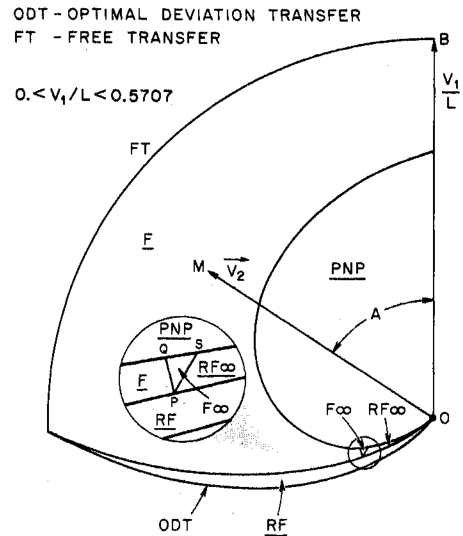


Fig. 15 The optimal transfers for $V_I/L = 0.435$, $A < \alpha_I + \alpha_2$. The curve PQ belongs to curve DEF of Fig. 14.

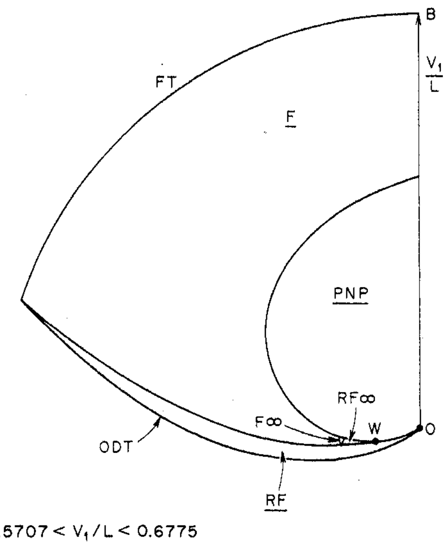


Fig. 16 The optimal transfers for $V_I/L = 0.6$, $A < \alpha_I + \alpha_2$. The point W belongs to the curve FO of Fig. 14. The region RF∞ does not extend to the point 0.

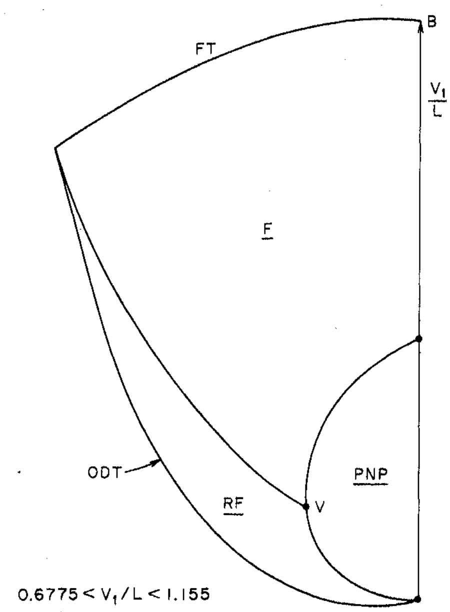


Fig. 17 The optimal transfers for $V_I/L = 1$, $A < \alpha_I + \alpha_2$. The point V belongs to the curve FNH of Fig. 14. The regions F∞ and RF∞ disappear.

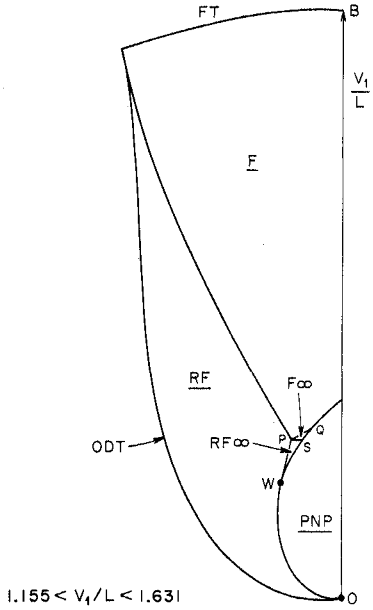


Fig. 18 The optimal transfers for $V_I/L = 1.45$, $A < \alpha_I + \alpha_2$. The curve PQ belongs to the curve HIJ, the point S to the curve HY, and W to the curve HZ of Fig. 14. The regions F∞ and RF∞ reappear.

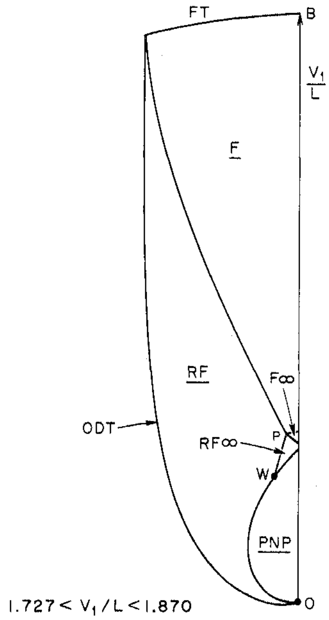


Fig. 20 The optimal transfers for $V_I/L = 1.8$, $A < \alpha_I + \alpha_2$. The region RF∞ reaches the straight line $A = 0$.

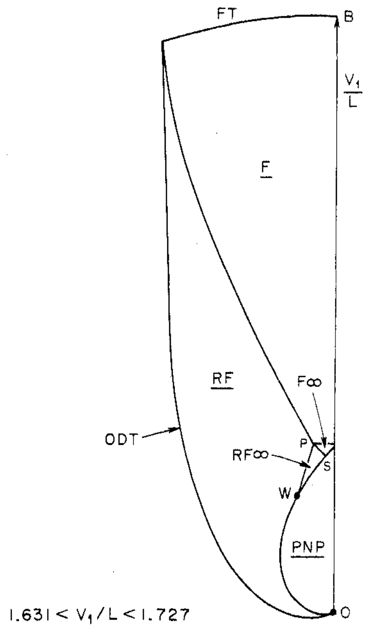


Fig. 19 The optimal transfers for $V_I/L = 1.7$, $A < \alpha_I + \alpha_2$. The region F∞ reaches the straight line $A = 0$.

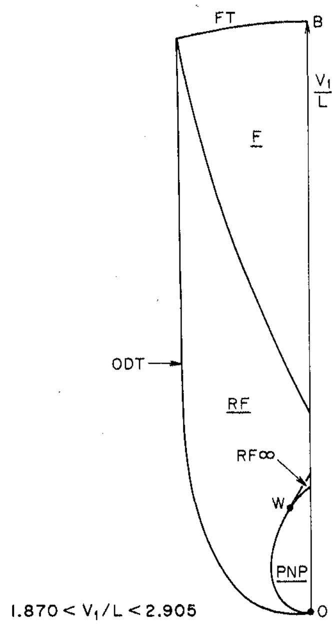


Fig. 21 The optimal transfers for $V_I/L = 2$, $A < \alpha_I + \alpha_2$. The region F∞ disappears. The region RF reaches the straight line $A = 0$.

The direction of the impulse at a finite distance and at infinity is opposite to that of the type F∞. Thus to go from the F∞ type to the RF∞ type, one must multiply the primer vector by minus one.

Transfers of Type ∞ RF

This final type of transfer is the most complicated. Of course, if the impulse at infinity is suppressed one obtains an optimal transfer of the RF type between the velocity V_{I0} and the velocity V_2. This transfer defines, as usual, the angles Σ and Δ at I_2 (Fig. 12). Let us call $(\pi - \phi_I)$ the angle which locates the direction of the first impulse, measured as usual from the local forward horizontal and positive in the clockwise direction if the transfer is in the counterclockwise direction and conversely, (Fig. 12), and let us call X and Y the radial and circumferential components of the velocity of arrival V_I' at the second impulse I_2 (X and Y are both positive). By the theory of Lawden[7] the transfer must satisfy the following two equations:

$$V_{I0}\cos\phi_I = \frac{V_2\cos^3\Delta\sin\Delta}{2\cos\Sigma\sin(\Sigma-\Delta)} \tag{19}$$

$$\frac{Y\cos\Delta}{U_{I0}^2} + \frac{\sin\Delta}{X}\left(I - \frac{Y^2}{U_{I0}^2}\right) = \frac{2L^2 Y\cos\phi_I}{U_{I0}^2(YU_{I0} + V_{I0}^2 - XV_{I0})} - \frac{\sin\phi_I}{V_{I0}} \tag{20}$$

Thus any RF transfers which satisfy these equations satisfy the optimality requirements of the ∞RF transfer. The optimality also requires that $0° < \phi_I <$ arcsin $\sqrt{3}/3 = 35.264°$.

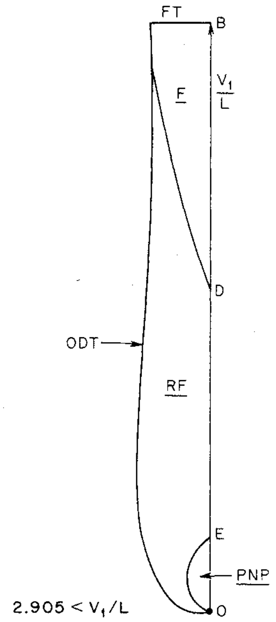


Fig. 22 The optimal transfers for $V_l/L = 3$, $A < \alpha_l + \alpha_2$. The region RF∞ disappears. When V_l goes to infinity, BD increases and goes to $2L$. OE decreases and goes to $0.34589L$.

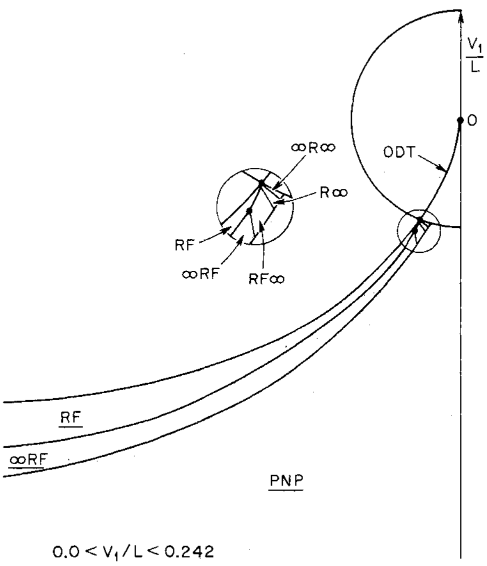


Fig. 23 The optimal transfers for $V_l/L = 0.1$, $A > \alpha_l + \alpha_2$. The region ∞RF and RF∞ intersect.

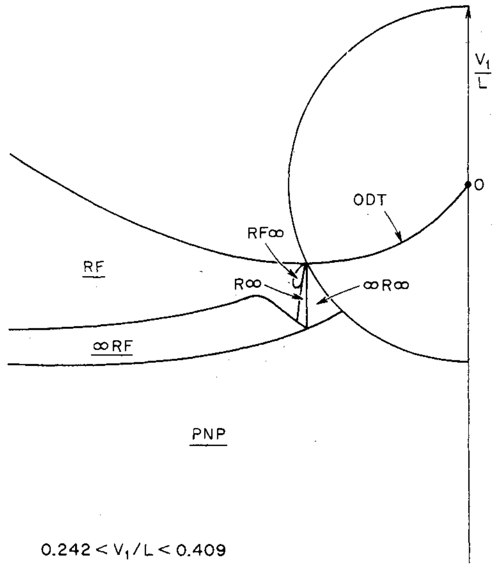


Fig. 24 The optimal transfers for $V_l/L = 0.3$, $A > \alpha_l + \alpha_2$. The region ∞RF and RF∞ do not intersect. The region ∞RD intersects both the R∞ and ∞R∞ regions.

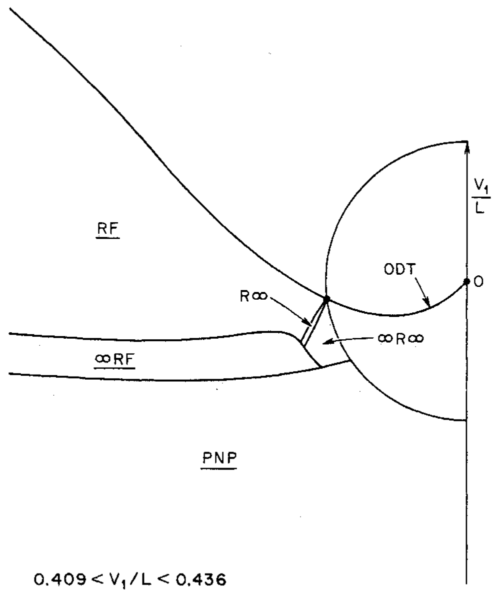


Fig. 25 The optimal transfers for $V_l/L = 0.42$, $A > \alpha_l + \alpha_2$. The region RF∞ disappears.

Results

The type of transfer which is optimal for a particular problem depends on three parameters. These may conveniently be taken as A, V_2/V_l, and V_l/L. Thus, to indicate the type of transfer which is optimal for any possible condition, a three-space must be partitioned. This is perhaps easiest to do by presenting various sections of the three-space. It is informative first to present a general view for the less than optimal deviation angle transfer in which the dependency on L is suppressed by projecting the space on the V_l, V_2 plane (Fig. 13). The parameters are reduced to A and V_2/V_l. An enlargement of the central portion is given (Fig. 14). On these plots most of the area is covered by sections I and IV, containing only single impulse optimal transfers and the PNP, or biparabolic, type of optimal transfers. Since these are precisely the two types of transfer which Gobetz[9] considered in his excellent study, this serves to somewhat justify his restrictions. Since L is suppressed here (Figs. 13 and 14) and may take on any value, the angle A may still approach 180° in particular cases. The entire problem space is collapsed into the half-circle shown. Note that the complete discussion of optimality is particularly easy if M is in the region I. Construct the optimal F transfer leading from V_l to V_2. If that transfer intersects the surface of the planet, the true optimal type is RF, if not, then compare type F $[C = (V_l + V_2) \sin\Delta]$ with type PNP $(C = U_l + U_2 - 2L)$ to determine the optimal transfer.

In the remaining figures (Figs. 15-30) the dependency on L is illustrated for both the less than and greater than optimal deviation angle transfers. Figures 15-22 are for the less than optimal deviation angle transfer and Figs. 23-30 are for the greater than optimal deviation angle transfers. These are sections of the problem space cut parallel to the V_l, V_2 plane at various values of V_2/L. On every figure the range of values of V_l/L for which the general characteristics of the partitioning are valid is given along with the specific value of V_l/L chosen to illustrate the area.

For a specific problem, the value V_l/L is calculated, and

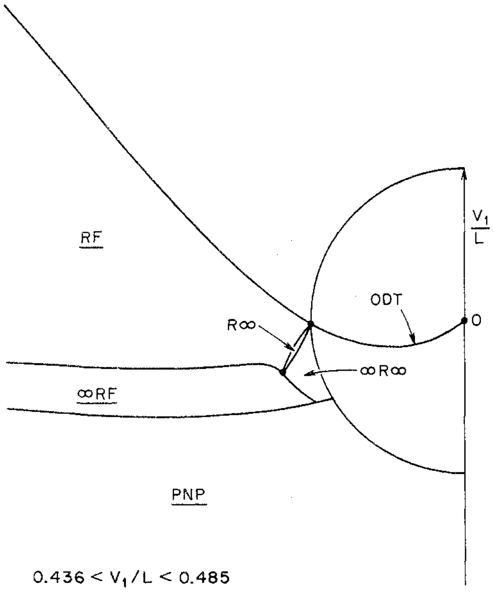


Fig. 26 The optimal transfers for $V_I/L = 0.45$, $A > \alpha_I + \alpha_2$. The ∞RF transfer intersects the intersection of the RF—R∞—∞R∞ regions.

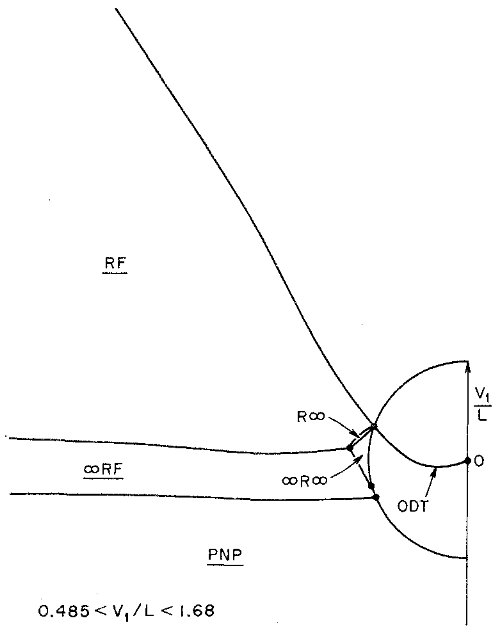


Fig. 27 The optimal transfer for $V_I/L = 0.60$, $A > \alpha_I + \alpha_2$. The ∞RF region reaches the boundary $V_2 = V_I$.

the proper figure chosen. The angle A between the two given asymptotes, and the ratio V_2/V_I determine the point on the figure corresponding to the given problem. From the figure, first it is easily determined if the desired turn angle is less than the optimal deviation turn angle, as is usually the case for feasible swingby missions. Second, the type of transfer which satisfies the maximum principle and has the minimum characteristic velocity is given. If the value V_I/L does not exactly correspond to that of the figure, and the point is near a partition line, then interpolation may be desirable.

Conclusions

This study considers in detail the types of transfers between given asymptotic terminal velocity vectors which provide the minimum characteristic velocity maneuver. The type of maneuver is not restricted in any way other than requiring that the trajectory always remain outside a sphere of radius R

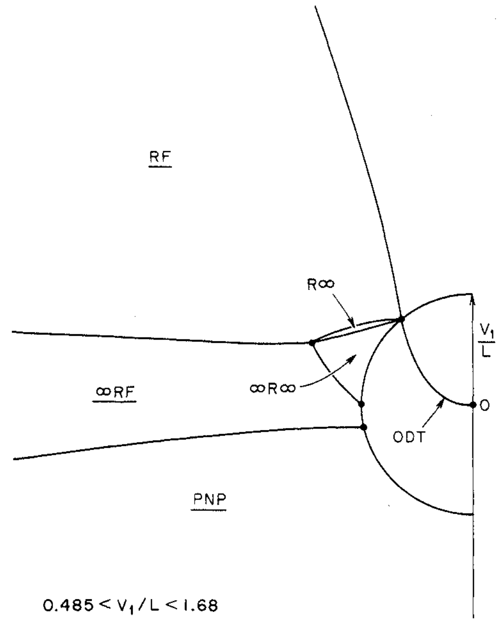


Fig. 28 The optimal transfer for $V_I/L = 1.0$, $A > \alpha_I + \alpha_2$. The R∞ regions begins to grow.

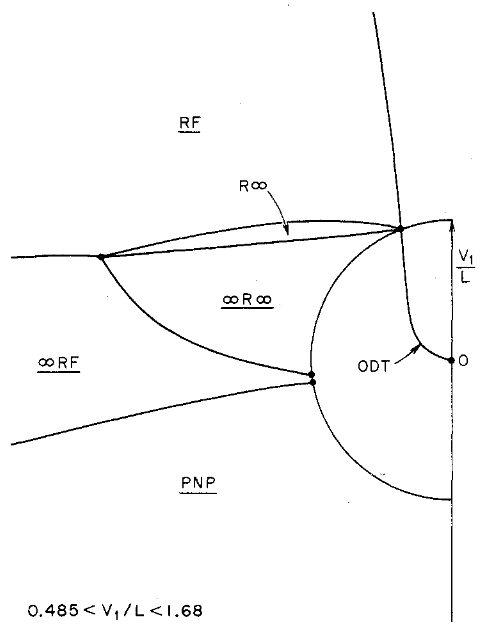


Fig. 29 The optimal transfer for $V_I/L = 1.5$, $A > \alpha_I + \alpha_2$. The region where ∞RF reaches the boundary where $V_2 = V_I$ begins to diminish.

surrounding the planet. The emphasis is on determining the type of transfer which provides the optimal transfer, and not necessarily in computing the details of the transfer. This is accomplished by partitioning the parameter three-space and displaying the results graphically.

Several general conclusions may be stated. Many of the results of the elliptical orbit transfer problem,[10,11] are found to hold here. In particular, with unlimited thrust available the optimal transfer, for all practical purposes, is always achieved by velocity impulses. Continuous thrust optimals appear only as academic singularities. Also, these impulses are constrained to lie in a very narrow band (the order of one degree in width) roughly midway between the local horizon and the tangent.

The largest portion of the boundary condition space for the less than optimal deviation angle transfer is covered by single impulse transfers. The second most common type is the PNP,

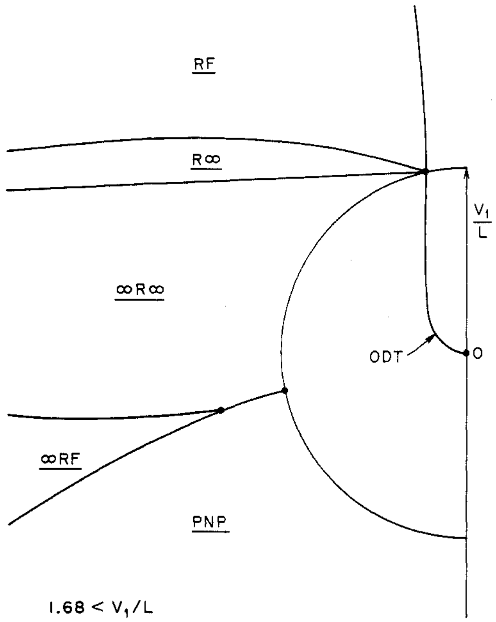


Fig. 30 The optimal transfer for $V_1/L=2.0$, $A>\alpha_1+\alpha_2$. The region where ∞RF reaches the boundary where $V_2=V_1$ disappears.

or biparabolic, transfer. For the greater than optimal deviation angle transfer space all transfers are grazing and the largest portion of the boundary condition space is covered by the PNP transfer. The second most common transfer is the single impulse transfer (RF, R∞). It should be noted that the two impulse transfers play a more important role than in the less than optimal deviation angle transfers. In any case, the optimal transfer never contains more than two finite impulses. Normally, every maneuver contains the infinitesimal impulse at infinity as the vehicle approaches the planet, to adjust the periapsis location. The only other time infinitesimal impulses appear is in the PNP transfer, which uses two additional infinitesimal impulses at infinity between its two finite impulses.

Finally, a general rule holds for the less than optimal deviation turn angle transfers. An impulse before periapse invariably increases velocity, while an impulse after periapse invariably decreases velocity. The contrary is true for the greater than optimal deviation turn angle transfers. (It must be borne in mind that a convention for direction of travel has been assumed in each case.)

An interesting extension of this problem which can be done without too much effort is to solve for the optimal swingby maneuver for some of the actual missions which are planned or are being proposed. Of course other mission requirements constrain the swingby, but the optimal transfer provides an absolute lower bound on the cost. In particular, this approach is proving profitable for Jupiter moon missions and outer planet flyby missions.

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